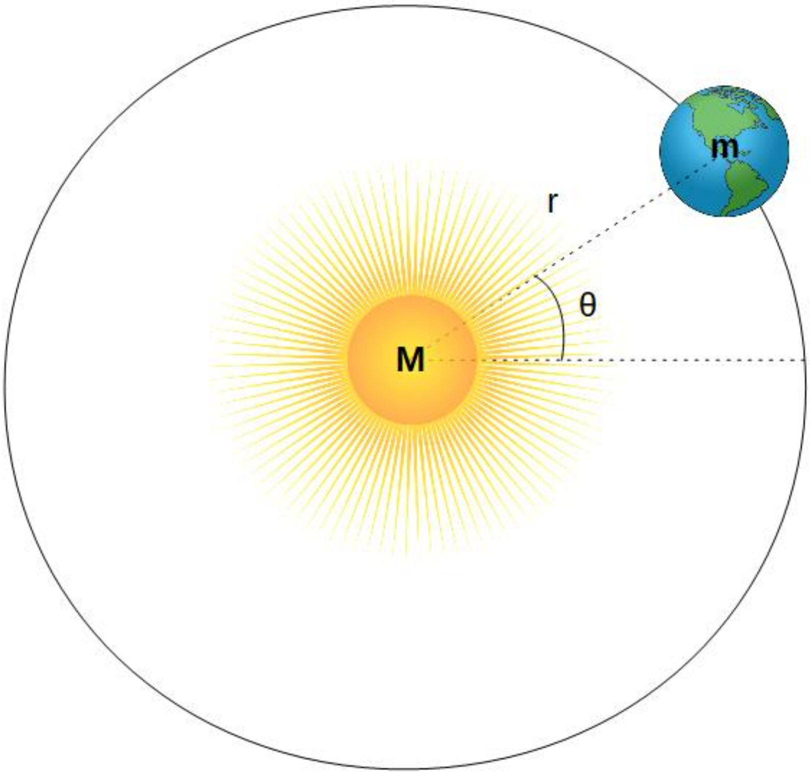
Example Problem #1

- **Find the equations of motion for the Earth revolving around the sun using Hamiltonian mechanics.** (Note; assume a circular orbit.)
- **Prove that the angular momentum of the Earth is conserved.**

Solution

The first thing to do is to define a set of convenient coordinates that describe the system. Note that you may have lots of different choices here, but I have chosen to go by according to this picture:



The position of Earth can be described like this:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

The key thing to note here is that both r and θ change with time (the magnitude of r stays constant, but it is a position vector, so the direction changes!)

Therefore we have to use the product rule to get the time derivatives of the position coordinates (velocities) to be:

$$\dot{x} = \dot{r}\cos\theta - \dot{\theta}r\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + \dot{\theta}r\cos\theta$$

The total kinetic energy for Earth is simply: $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$

Then inserting $\dot{x}$ and $\dot{y}$, we get:

$$T = \frac{1}{2}m\left(\dot{r}^2\cos^2\theta - 2\dot{r}\dot{\theta}r\sin\theta\cos\theta + \dot{\theta}^2 r^2\sin^2\theta + \dot{r}^2\sin^2\theta + 2\dot{r}\dot{\theta}r\sin\theta\cos\theta + \dot{\theta}^2 r^2\cos^2\theta\right)$$

Looking at this expression a bit closer, we can indeed see that it simplifies to a much nicer form:

$$T = \frac{1}{2}m\left(\dot{r}^2\cos^2\theta - 2\dot{r}\dot{\theta}r\sin\theta\cos\theta + \dot{\theta}^2 r^2\sin^2\theta + \dot{r}^2\sin^2\theta + 2\dot{r}\dot{\theta}r\sin\theta\cos\theta + \dot{\theta}^2 r^2\cos^2\theta\right)$$

$\sin^2\theta + \cos^2\theta = 1$

These terms cancel each other

$\sin^2\theta + \cos^2\theta = 1$

After these simplifications we're just left with:

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

The Earth also has a potential energy under the influence of the Sun, namely gravitational potential energy in the form:

$$V = -\frac{GmM}{r}$$

The Lagrangian for Earth is therefore:

$$\boxed{L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \left(-\frac{GmM}{r}\right)}$$

The next thing to do is to define the canonical momenta for both of our generalized coordinates (r and θ) through the Lagrangian:

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad\qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

Then solving them both for the velocities, we get:

$$\dot{r} = \frac{p_r}{m} \qquad\qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Remember the general form of the Hamiltonian:

$$H = \sum_i p_i \dot{q}_i - L$$

The Hamiltonian for our system is:

$$H = \dot{\theta} p_\theta + \dot{r} p_r - L$$

And by inserting the velocities and the equation for the Lagrangian:

$$H = \frac{p_\theta}{mr^2} p_\theta + \frac{p_r}{m} p_r - \frac{1}{2} m \left(\frac{p_r}{m}\right)^2 - \frac{1}{2} mr^2 \left(\frac{p_\theta}{mr^2}\right)^2 - \frac{GmM}{r}$$

$$H = \frac{p_\theta^2}{mr^2} + \frac{p_r^2}{m} - \frac{1}{2}\frac{p_r^2}{m} - \frac{1}{2}\frac{p_\theta^2}{mr^2} - \frac{GmM}{r}$$

$$H = \frac{p_\theta^2}{2mr^2} + \frac{p_r^2}{2m} - \frac{GmM}{r}$$

Now we can actually use Hamilton's equations to find the equations of motion for the Earth (we only need the first equations since we already derived the velocities above).

Here we also get an equation for
both of the momenta, the first one for p_r being:

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{GmM}{r^2}$$

Hamilton's equations of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

This particular equation gives us the time derivative of the momentum (i.e. force in Newtonian mechanics) associated with the variable r. In analogy to Newtonian mechanics, it corresponds to **the gravitational force** between the Sun and the Earth.

You might notice the second term being just the regular gravitational force, however Hamilton's equations give us something else too. The first term is actually **the centrifugal force**, which is a fictitious force that only appears to be there in the reference frame of the object itself.

Moving on to the momentum associated with θ, we get something interesting there as well. Since the Hamiltonian, interestingly enough does not depend on θ at all, we get from Hamilton's equations:

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

Now think about what this tells us. The momentum associated with θ is a momentum associated with rotating an angle or more simply put, **the angular momentum**. On the other hand, the time derivative of something being equal to 0 means that something *does not change with time* (i.e. it stays constant). Therefore, this equation indeed proves **the conservation of angular momentum** for the Earth.

Example Problem #2

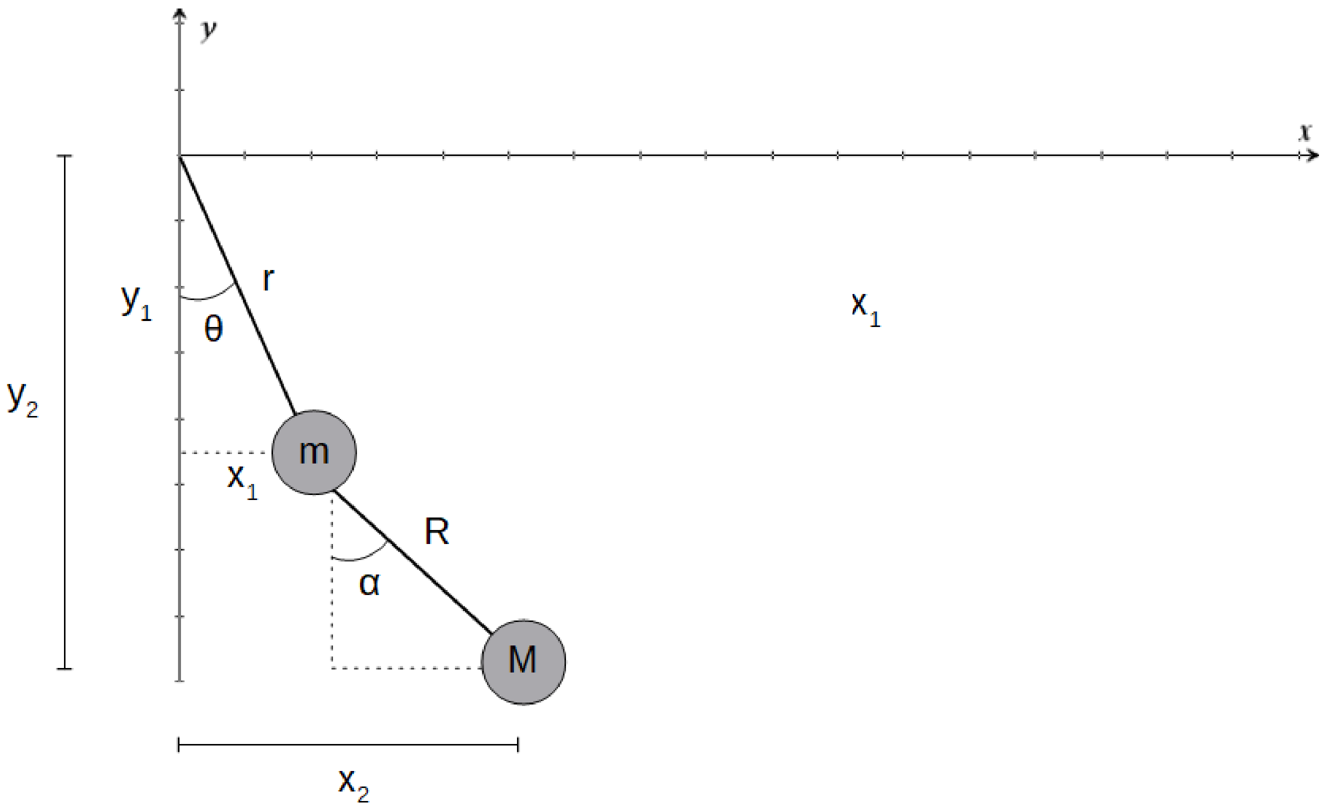
- **Find the Hamiltonian for a double pendulum swinging in a gravitational field. Express the Hamiltonian in terms of the momenta.**

Hint; use a Cartesian coordinate system and try to express the coordinates in terms of *angles*. The rods are also assumed to have no mass.

Solution

First, we're going to define a set of generalized coordinates for the system, in particular by placing the pendulum in a Cartesian coordinate system (note; there are also other useful coordinates, but we'll choose the regular Cartesian system).

The pendulum is placed so that the start of the first rod is at the origin. All of the parameters we'll need are as follows:



It is again useful to describe the positions of both of the pendulum bobs by the angles as shown in the picture. For the first bob, the x and y -coordinates, expressed in terms of the angle θ, are as follows:

$$x_1 = r\sin\theta$$

$$y_1 = -r\cos\theta$$

(note that y_1 has a minus-sign because it is measured downwards from the origin)

The coordinates for the second bob, on the other hand are:

$$\begin{aligned} x_2 &= x_1 + R\sin\alpha \\ &= r\sin\theta + R\sin\alpha \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 - R\cos\alpha \\ &= -r\cos\theta - R\cos\alpha \end{aligned}$$

The velocities for each of the coordinates are (note that are r and R are simply the rod lengths, i.e. constants):

$$\dot{x}_1 = r\dot{\theta}\cos\theta \qquad \dot{x}_2 = r\dot{\theta}\cos\theta + R\dot{\alpha}\cos\alpha$$

$$\dot{y}_1 = r\dot{\theta}\sin\theta \qquad \dot{y}_2 = r\dot{\theta}\sin\theta + R\dot{\alpha}\sin\alpha$$

Now, the kinetic energy for the first bob is:

$$\begin{aligned} T_1 &= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) \\ &= \frac{1}{2}m\left(r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta\right) \end{aligned}$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$= \frac{1}{2}mr^2\dot{\theta}^2$$

The kinetic energy for the second bob is a little bit more complicated:

$$\begin{aligned} T_2 &= \frac{1}{2}M\left(\dot{x}_2^2 + \dot{y}_2^2\right) \\ &= \frac{1}{2}M\left(\left(r\dot{\theta}\cos\theta + R\dot{\alpha}\cos\alpha\right)^2 + \left(r\dot{\theta}\sin\theta + R\dot{\alpha}\sin\alpha\right)^2\right) \\ &= \frac{1}{2}M\left(r^2\dot{\theta}^2\cos^2\theta + 2rR\dot{\alpha}\dot{\theta}\cos\theta\cos\alpha + R^2\dot{\alpha}^2\cos^2\alpha \right. \\ &\qquad \left. + r^2\dot{\theta}^2\sin^2\theta + 2rR\dot{\alpha}\dot{\theta}\sin\theta\sin\alpha + R^2\dot{\alpha}^2\sin^2\alpha\right) \end{aligned}$$

After expanding those, we can use some of the rules for trigonometric functions to simplify the expression:

$$T_2 = \frac{1}{2}M\left(r^2\dot{\theta}^2\cos^2\theta + 2rR\dot{\alpha}\dot{\theta}\cos\theta\cos\alpha + R^2\dot{\alpha}^2\cos^2\alpha\right.$$

$$\left. + \, r^2\dot{\theta}^2\sin^2\theta + 2rR\dot{\alpha}\dot{\theta}\sin\theta\sin\alpha + R^2\dot{\alpha}^2\sin^2\alpha\right)$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos\theta\cos\alpha + \sin\theta\sin\alpha = \cos(\theta - \alpha)$$

$$\cos^2\alpha + \sin^2\alpha = 1$$

$$T_2 = \frac{1}{2}M\left(r^2\dot{\theta}^2 + R^2\dot{\alpha}^2 + 2rR\dot{\alpha}\dot{\theta}\cos(\theta - \alpha)\right)$$

The total kinetic energy of the system is then the sum of the kinetic energies of both bobs:

$$T = T_1 + T_2 = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}M\left(r^2\dot{\theta}^2 + R^2\dot{\alpha}^2 + 2rR\dot{\alpha}\dot{\theta}\cos(\theta - \alpha)\right)$$

The double pendulum is also in a gravitational field, meaning that both of the bobs have a gravitational potential energy, which depends on the height of the bob (the y-coordinate):

$$V_1 = mgy_1 = -mgr\cos\theta$$

$$V_2 = Mgy_2 = -Mg\left(r\cos\theta + R\cos\alpha\right)$$

The total potential energy of the system is also simply the sum of these two:

$$V = V_1 + V_2 = -mgr\cos\theta - Mg\left(r\cos\theta + R\cos\alpha\right)$$

The Lagrangian of the system is then:

$$L = T - V$$

$$= \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}M\left(r^2\dot{\theta}^2 + R^2\dot{\alpha}^2 + 2rR\dot{\alpha}\dot{\theta}\cos(\theta - \alpha)\right)$$

$$+mgr\cos\theta + Mg\left(r\cos\theta + R\cos\alpha\right)$$

Remember the steps for finding a Hamiltonian:

1. Construct the Lagrangian for the system through a set of generalized coordinates.
2. Find the canonical momenta from the Lagrangian.
3. Solve for the velocity from the canonical momentum equation.
4. Insert the velocity term in the general form of the Hamiltonian (to replace velocity with momentum).
5. Simplify the Hamiltonian if needed.

We can now obtain the canonical momenta associated with the angles θ and α, which are:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} + Mr^2\dot{\theta} + MrR\dot{\alpha}\cos(\theta - \alpha)$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = MR^2\dot{\alpha} + MrR\dot{\theta}\cos(\theta - \alpha)$$

And rearranging them, we get:

$$\dot{\theta} = \frac{p_\theta - MrR\dot{\alpha}\cos(\theta - \alpha)}{(m + M)\,r^2}$$

$$\dot{\alpha} = \frac{p_\alpha - MrR\dot{\theta}\cos(\theta - \alpha)}{MR^2}$$

Now as you can see, the only problem is that both of these equations include both of the velocity coordinates. Luckily, in this situation there are two equations with two variables we wish to solve for, so we get a system of equations:

$$\begin{cases} \dot{\theta} = \dfrac{p_\theta - MrR\dot{\alpha}\cos(\theta-\alpha)}{(m+M)r^2} \\ \dot{\alpha} = \dfrac{p_\alpha - MrR\dot{\theta}\cos(\theta-\alpha)}{MR^2} \end{cases}$$

Systems of equations like this are pretty simple to solve. Just insert either one of the equations into the other and solve for the variable you wish to. Then do the same for the other equation.

After doing that and a fair bit of very tedious algebra, we arrive at these two equations:

$$\dot{\theta} = \frac{p_\theta R - p_\alpha r\cos(\theta-\alpha)}{(m+M)\,r^2 R - Mr^2 R\cos^2(\theta-\alpha)}$$

$$\dot{\alpha} = \frac{(m+M)\,p_\alpha r - p_\alpha Mr\cos(\theta-\alpha) - p_\theta MR\cos(\theta-\alpha) + p_\alpha Mr\cos(\theta-\alpha)}{(m+M)\,MR^2 r - M^2 R^2 r\cos(\theta-\alpha)}$$

We are almost done here. As we now have the velocities, we can then construct the Hamiltonian for this system, which is:

$$\begin{aligned} H &= \dot{\theta}p_\theta + \dot{\alpha}p_\alpha - L \\ &= \dot{\theta}p_\theta + \dot{\alpha}p_\alpha - \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}M\left(r^2\dot{\theta}^2 + R^2\dot{\alpha}^2 + 2rR\dot{\alpha}\dot{\theta}\cos(\theta-\alpha)\right) \\ &\quad - mgr\cos\theta - Mg\,(r\cos\theta + R\cos\alpha) \end{aligned}$$

Now it's only a matter of inserting the velocity equations into this Hamiltonian. This is going to get pretty ugly, but by doing it, we get possibly the longest equation ever:

$$
\begin{aligned}
H = &\frac{p_\theta R - p_\alpha r\cos(\theta-\alpha)}{(m+M)\,r^2R - Mr^2R\cos^2(\theta-\alpha)}p_\theta \\
&+\frac{(m+M)\,p_\alpha r - p_\alpha Mr\cos(\theta-\alpha) - p_\theta MR\cos(\theta-\alpha) + p_\alpha Mr\cos(\theta-\alpha)}{(m+M)\,MR^2r - M^2R^2r\cos(\theta-\alpha)}p_\alpha \\
&-\frac{1}{2}(m+M)\,r^2\left(\frac{p_\theta R - p_\alpha r\cos(\theta-\alpha)}{(m+M)\,r^2R - Mr^2R\cos^2(\theta-\alpha)}\right)^2 \\
&-\frac{1}{2}MR^2\left(\frac{(m+M)\,p_\alpha r - p_\alpha Mr\cos(\theta-\alpha) - p_\theta MR\cos(\theta-\alpha) + p_\alpha Mr\cos(\theta-\alpha)}{(m+M)\,MR^2r - M^2R^2r\cos(\theta-\alpha)}\right)^2 \\
&-MRr\cos(\theta-\alpha)\,\frac{(p_\theta R - p_\alpha r\cos(\theta-\alpha))\,((m+M)\,p_\alpha r - p_\alpha Mr\cos(\theta-\alpha) - p_\theta MR\cos(\theta-\alpha) + p_\alpha Mr\cos(\theta-\alpha))}{((m+M)\,r^2R - Mr^2R\cos^2(\theta-\alpha))\,(m+M)\,MR^2r - M^2R^2r\cos(\theta-\alpha)} \\
&-mgr\cos\theta - Mg\,(r\cos\theta + R\cos\alpha)
\end{aligned}
$$

This equation looks quite complex, so it might be easier to digest in pieces. We can also see how it compares to something much simpler, say a system with only one particle moving in the x-direction:

$$
\begin{aligned}
H = &\underbrace{\frac{p_\theta R - p_\alpha r\cos(\theta-\alpha)}{(m+M)\,r^2R - Mr^2R\cos^2(\theta-\alpha)}p_\theta} \\
&\underbrace{+\frac{(m+M)\,p_\alpha r - p_\alpha Mr\cos(\theta-\alpha) - p_\theta MR\cos(\theta-\alpha) + p_\alpha Mr\cos(\theta-\alpha)}{(m+M)\,MR^2r - M^2R^2r\cos(\theta-\alpha)}p_\alpha} \\
&\underbrace{-\frac{1}{2}(m+M)\,r^2\left(\frac{p_\theta R - p_\alpha r\cos(\theta-\alpha)}{(m+M)\,r^2R - Mr^2R\cos^2(\theta-\alpha)}\right)^2} \\
&\underbrace{-\frac{1}{2}MR^2\left(\frac{(m+M)\,p_\alpha r - p_\alpha Mr\cos(\theta-\alpha) - p_\theta MR\cos(\theta-\alpha) + p_\alpha Mr\cos(\theta-\alpha)}{(m+M)\,MR^2r - M^2R^2r\cos(\theta-\alpha)}\right)^2} \\
&\underbrace{-MRr\cos(\theta-\alpha)\,\frac{(p_\theta R - p_\alpha r\cos(\theta-\alpha))\,((m+M)\,p_\alpha r - p_\alpha Mr\cos(\theta-\alpha) - p_\theta MR\cos(\theta-\alpha) + p_\alpha Mr\cos(\theta-\alpha))}{((m+M)\,r^2R - Mr^2R\cos^2(\theta-\alpha))\,(m+M)\,MR^2r - M^2R^2r\cos(\theta-\alpha)}} \\
&\underbrace{-mgr\cos\theta - Mg\,(r\cos\theta + R\cos\alpha)}
\end{aligned}
$$

The red terms are simply the generalized velocities for both of the angles θ and α (multiplied by the angular momentum associated with each if them). This is analogous to the following term in the Hamiltonian for the single-particle system (except in the above one we just have two of these terms):

$$H = \underbrace{\dot{x}}p - L = \underbrace{\frac{p}{m}}p - L$$

All of the terms after the red ones come from the Lagrangian. The blue parts are analogous to the ”normal” kinetic energy terms you’d typically expect in the Lagrangian, which are in the form of $\frac{1}{2}mv^2$. In the Hamiltonian for the simple system, it is these terms (after inserting the Lagrangian):

$$H = \frac{p^2}{m} - \underbrace{\frac{1}{2}m\dot{x}^2} + V(x) = \frac{p^2}{m} - \underbrace{\frac{1}{2}m\left(\frac{p}{m}\right)^2} + V(x)$$

The orange part, on the other hand, is kind of interesting. It is actually an extra term that comes from the fact that **the system has two particles with one of them being dependent on the other** (the position of the second bob depends on the position of the first one as they are attached). This term does not have any particular analogous counterpart in the single-particle system.

The green part is simply both of the bobs’ potential energies, which can easily be seen from the fact that these two terms are not dependent on the momenta or the velocities, only **the generalized positions** (the angles θ and α). This is exactly what is typical for potential energies, only depend on position rather than motion.

The red, blue and orange terms together form the kinetic energy part of the Hamiltonian, while the green terms cover the potential energy part. In the simple system case, this is analogous to:

$$H = \underbrace{\frac{p^2}{2m}}_{\text{The kinetic energy}} + \underbrace{V(x)}_{\text{The potential energy}}$$

There is one more thing worth noting about the double pendulum. If you've ever seen the motion of a double pendulum system, it seems extremely chaotic and unpredictable. While based on this example, it would theoretically be possible to predict the motion of a double pendulum, in practice this is not really the case.

For example, if we wanted to actually predict its motion for even a couple of minutes, even that would require insanely accurate measurements about the initial conditions of the system. Generally, for any real world systems, the longer the duration we want to get accurate mathematical predictions for, the more accurate our knowledge of the initial conditions have to be.

Example Problem #3

- **Prove the following relationship between the Hamiltonian and the Lagrangian:**

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Hint; construct a Lagrangian that is *explicitly dependent on time.*

- **Show that energy is conserved if the Lagrangian does NOT explicitly depend on time.**

Solution

This problem is actually quite simple, at least mathematically. You just have to know where to start, which is by constructing an *explicitly* time-dependent Lagrangian.

Now, the Lagrangian is typically *implicitly* dependent on time, which just means that it is dependent on time, but not directly, only through the position and velocity variables. What we want now is a Lagrangian that is directly dependent on time (as well as the position and velocity):

$$L = L\left(q_i\ ,\ \dot{q}_i\ ,\ t\right)$$

Now it’s only a matter of looking at how the Hamiltonian varies with a Lagrangian like this. That is pretty simple. It’s the exact same process we did when deriving Hamilton’s equations of motion, so feel free to go back to the article at profoundphysics.com and read through that part of you have to. This time there is just one more variable included, which will give us a third (but not so commonly used) equation of motion. The Hamiltonian in this case is (leaving out the summation sign):

$$H = \dot{q}_i p_i - L\left(q_i\ ,\ \dot{q}_i\ ,\ t\right)$$

Then, looking at the variance in the Hamiltonian, we get:

$$dH = \dot{q}_i dp_i + p_i d\dot{q}_i - \left(\frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt\right)$$

Notice that this is the exact same thing we had when deriving the equations of motion, except for the last part (the term with time), which comes from the explicitly time-dependent Lagrangian.

Here we can also get some terms to cancel by recalling the definition of the **canonical momentum** and inserting it:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$dH = \dot{q}_i dp_i + p_i d\dot{q}_i - \left(\frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial L}{\partial t}dt\right)$$

These terms
cancel out and
we’re left with:

$$dH = \dot{q}_i dp_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt$$

Now, comparing the above expression to the **general chain rule** for varying the Hamiltonian if we only know its variables (this time there is also the explicit time variable):

$$dH\left(q_i\ ,\ p_i\ ,\ t\right) = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt$$

And for the mathematics to hold consistent, we have to have these terms equal each other:

$$dH = \dot{q}_i dp_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial t}dt$$

$$dH = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt$$

The grey arrows here will just give us the usual Hamiltonian equations of motion. What we're really interested in is the red one, which gives us:

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Now, the interesting thing here is not so much how we derived or proved this equation, it is about what this equation tells us; given that the Lagrangian is explicitly dependent on time, the Hamiltonian will also be explicitly dependent on time. The two only differ by a minus-sign, but the physical implications of this is that **they change opposite to one another with time**. This means that if the value of the Lagrangian was to *decrease* with time, the Hamiltonian would *increase* with time, and vice versa.

Another interesting implication of this equation is **energy conservation**. In particular, the energy of a system is conserved if the Lagrangian is NOT explicitly time-dependent. This is easy to prove with the above equation. The Lagrangian being not explicitly dependent on time means that the Lagrangian is only a function of the position and velocity coordinates:

$$L = L\left(q_i\ ,\ \dot{q}_i\right)$$

Now, as the Lagrangian does not have a direct time variable, the partial derivative with respect to time is simply 0:

$$\frac{\partial H}{\partial t} = -\frac{\partial}{\partial t} L\left(q_i\ ,\ \dot{q}_i\right) = 0$$

This obviously tells us that the Hamiltonian (total energy) **does not change with time**. Therefore, energy conservation is proven IF the Lagrangian is not explicitly time-dependent.